RENORMGROUP INVARIANTS AND APPROXIMATIONS OF MAPPINGS

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(Dated: November 18, 2004)

The relationship between mappings of sets and renormalization group transformations is established, and renormalization group invariants of such mappings are found. These results are valid both for continuous and discrete mappings and for various dimensionality of image and preimage of the mappings too. It is suggested a number of mapping approximations improved in comparison with an ordinary power expansion. The approximations take into account the global one-to-one character of the mappings. The method is illustrated by a number of examples: by reconstructing of some analytical functions, calculating the integral of the typical partition function of statistical mechanics and the ground state energy for the quartic anharmonic oscillator. In the whole range of nonlinearity parameter varying from zero up to infinity the accuracy of the RG approximation based on a few terms of divergent series is about 0.06% in the next to last case and 0.004% in the last one.

PACS numbers: 02.70.-c; 02.30.-f; 02.30.Mv; 05.20.-y; 01.90.+g

I. INTRODUCTION

Renormalization group (RG) conception originally arose in the beginning of the fifties [1] as a result of the discovery of special group of continuous transformations in quantum electrodynamics. This transformations are connected with a complicated procedure of renormalization, that is 'removing of ultra-violet infinities'. It turned out, that physical quantities do not vary at simultaneous rescaling of the 4-momentum transfer squared z, mass squared y, and also special transformation of the dimensionless charge squared g ($g \to \overline{g}(z, y, g)$). The invariant charge $\overline{g}(z, y, g)$ obeys to the equation [2, 3]

$$\overline{g}(z, y, g) = \overline{g}(z/t, y/t, \overline{g}(t, y, g)). \tag{1}$$

For the massless quantum-field model instead of (1) the following relation is valid

$$\overline{g}(z, g) = \overline{g}(z/t, \overline{g}(t, g)). \tag{2}$$

The generalization of equation (2) for the two-charge quantum-field model was proposed also [3]. In this case, instead of expression (2) the following two relations are fulfilled

$$\overline{g}_i(z, g_1, g_2) = \overline{g}_i(z/t, \overline{g}_1(t, g_1, g_2), \overline{g}_2(t, g_1, g_2)) \qquad i = 1, 2 \tag{3}$$

The general solution of such functional equations was obtained [4].

The rapid expansion of the RG method far beyond the quantum-field theories began since the seventies. It was strongly promoted by the article [5]. In this article the conception of an approximate RG was introduced for analysis of thermodynamic systems behaviour near the phase transition points. Afterwards the RG method was applied for description of turbulence [6, 7], polymeric compounds [8], radiation transfer in opaque mediums with the strong frequency dependence of the quantum path length [9], fractals [10], scripts of the dynamic systems transitions to the determined chaos [11], and also was applied to other problem. The relation was found out between nonlinear problems of radiation transport and additive variant of the RG [12, 13]. This result was extended on the wide range of the physical problems; and the functional self-similarity concept, that generalizes the usual self-similarity concept, was introduced [14]. The modern state of the researches concerned with use of this concept for boundary-value problems of the mathematical physics is represented in the recent review [15].

In the present paper a relationship between mappings from one set into another one and RG transformations is established, and RG invariants of such mappings are found. On this basis a number of RG approximations that are founded on a few terms of series of one-to-one functions is constructed. They represent better approximations of the sought function in comparison with its Taylor series expansion because the RG-approximations inherits its one-to-one nature.

The necessity for reasonable approximations of physical quantities is connected to the fact that the majority of realistic problems cannot be solved exactly. To solve them, one uses various approximating methods that are in general different kinds of perturbation theories near a chosen zero approximation. These methods are evidently convenient if the perturbation (interaction or coupling) parameter g is small, $g \ll 1$. However, if $g \gtrsim 1$ the expansion in powers

of g have no sense. But for many realistic systems, because of their complexity, it is technically impossible to use other calculation techniques except for the perturbation one. In such a case there are a number of resummation (or reconstruction) methods that allow to find the value of some function $f(g)$ in the case of $g \gtrsim 1$ using only the results of its calculation in the limit $g \ll 1$. These methods include the improved perturbation theory, the Padé approximation, the Borel summation, the conformal mapping and their combinations [16, 17], the continued fraction approximations [18], and the Romberg algorithm [19]. These methods, to be accurate enough, need the calculation of many subsequent approximations for the sought function. But if system is so complex that we are able to calculate solely a couple of approximations, then the majority of these methods lose their applicability. In this situation the method of RG approximation seems especially relevant because of its quite good accuracy when using only a few terms of the perturbation theory.

II. RENORMGROUP INVARIANTS OF ONE-TO-ONE MAPPINGS

At first let us consider a function of a single variable $f(x)$ as a primary mapping. It is assumed hereinafter that the function $f(x)$ is one-to-one, i.e. it has the inverse function. Now we consider the value of this function at the argument $x_1 + x$. Let us express the quantity x via the inverse function: $x = f^{-1}(f(x))$. This relation allows one to write down $f(x_1 + x)$ as follows:

$$f(x_1 + x) = f(x_1 + f^{-1}(f(x))) \equiv F(x_1; f(x)). \tag{4}$$

Thus, we have presented $f(x_1 + x)$ as a new function F of two variables, value of the initial function f at the original argument x and the translation x_1. According to definition (4), the value of the function F at $x_1 = 0$ coincides with the value of the initial function f at the argument x:

$$F(0; f(x)) = f(x) \tag{5}$$

Now let us consider the value of the function f at the argument $\tilde{x} = x + x_1 + x_2$ displaced sequentially on x_1 and x_2 with respect to x. The value of the function $f(\tilde{x})$ can be represented via the function F by two equivalent ways. According to the one way, it is possible to consider the two sequential displacement x_1 and x_2 as a whole displacement $x_w = x_1 + x_2$ and to take advantage of the definition (4), that leads to the relation

$$f(x + (x_1 + x_2)) = F(x; f(x_1 + x_2)). \tag{6}$$

And then it is possible to take advantage of definition (4) once more to transform the function $f(x + x_2)$ on the right-hand side of expression (6) in terms of F:

$$F(x; f(x_1 + x_2)) = F(x; F(x_1; f(x_2))). \tag{7}$$

On the other way, at first one can use definition (4) with replacement $x + x_1$ for x, that results in

$$f((x + x_1) + x_2) = F(x + x_1; f(x_2)). \tag{8}$$

Equating the right parts of expressions (7) and (8), taking into account equality (6), we get the functional equation

$$F(x + x_1; f)) = F(x; F(x_1; f)). \tag{9}$$

Hereinafter we have omitted for brevity the argument x of the function f. Equation (9) coincides in its structure with the equation (1) for the radiation transport in [13].

Note that relation (9) turns into (2) by means of the replacements $x = \ln(z/t)$, $x_1 = \ln(t)$, and the obvious change of the notations: $f \to g$, $F \to \overline{g}$.

From equation (9) the partial differential equation for the function F is easily obtained [13, 14, 15]. For this purpose it is enough to differentiate both parts of equation (9) on $x_1 \to 0$ and to take advantage of property (5):

$$[\frac{\partial}{\partial x} - \beta(f)\frac{\partial}{\partial f}]F(x; f) \equiv RF(x; f) = 0; \tag{10}$$

$$\beta(f) = \frac{\partial F(x_1; f)}{\partial x_1}|_{x_1=0}. \tag{11}$$

Functional equation (9) expresses the invariance of F under the RG transformation [14, 15] representing a simultaneous one-parametric point transformation of both arguments of the function F:

$$T(x_1): \qquad x \to \tilde{x} = -x_1 + x, \qquad f \to \tilde{f} = F(x_1; f); \qquad F(0; f) = f. \tag{12}$$

Equation (9) guarantees fulfillment of the group property $T(x_1)T(x_2) = T(x_1 + x_2)$. Note that differential equation (10) represents the infinitesimal form of transformation (12), and R is the infinitesimal operator of RG-symmetry.

III. RENORMGROUP APPROXIMATIONS OF ONE-TO-ONE MAPPINGS

Frequently it is possible to define the behavior of some sought function only in a vicinity of some point. Such situation is typical for iterative procedures or for one of the basic methods of theoretical and mathematical physics, perturbation theory. As a rule, the calculation of the higher-order corrections is connected with significant difficulties of the technical or basic character (for example, because of the divergence of the higher-order corrections). It is known also that a finite series function approximation often describes wrongly the long range function behaviour. Relations (9), (10) allow to receive improved approximation of the sought function (so-called RG method [3]). This approximation more correctly describes the long range behavior of the sought functions. The solution of equation (10) with boundary condition (5) is possible to represent as

$$F(x;f) = X^{-1}(x + X(f)), \tag{13}$$

$$X(f) = \int \frac{df}{\beta(f)}\,, \tag{14}$$

where X^{-1} is inverse to X function. Ordinarily, to determine the function (14) one use an approximate value for β (11) calculated in the small parameter limit $f = 0$. Let us consider other approach that does not presuppose the smallness. According to definitions (11) and (5) we have:

$$\beta(f) = \frac{df}{dx} \equiv f^{'}(x). \tag{15}$$

Suppose that we know an approximate expression of $f(x)$ in a neighborhood of some point x_3, for example, as a Taylor series (without loosing of generality one can set $x_0 = 0$):

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots. \tag{16}$$

Series (16) can be inverted:

$$x = \phi + b_2\phi^2 + b_3\phi^3 + \ldots, \tag{17}$$

where $\phi = (f - a_0)/a_1$, the coefficients b_i are expressed via the coefficients of initial series (16). The sequential substitution of expansion (16) and (17) into the expression (15) enables us to get the required dependence of $\beta(f)$ as an infinite series on ϕ. If we use various finite series approximations of $\beta(f)$, we receive various RG approximations of the sought function by the inversion of expression (14). So, the linear approach of $\beta(f)$ gives the following RG approximation of the sought function $f(x)$:

$$f(x) \simeq f_{\beta}^{(2)}(x) = a_0 + \frac{{a_1}^2}{2a_2}\left[\exp(2a_2x/a_1) - 1\right]. \tag{18}$$

From the square-law approach of $\beta(f)$ another RG-approximation follows:

$$f(x) \simeq f_{\beta}^{(3)}(x) = a_1 + \frac{{a_1}^2}{a_2}\left[\frac{a_1}{a_2}\kappa\coth(\kappa x) - 1\right]^{-1}, \tag{19}$$

where $\kappa = \sqrt{3[(a_2/a_1)^2 - a_3/a_1]}$.

It is interesting to note, that expression (14) represents exactly the variable x as a function of f taking into account (15). Thus, expressions (14) and (17) are equivalent. Therefore, one can get one more variant of the RG function f approximation, using only a few sequential terms of series (17) and inverting the obtained approximate equation with respect to f. So, the square-law approach of series (17) results in the following RG approximation:

$$f(x) \simeq f_{X}^{(2)}(x) = a_0 + \frac{{a_1}^2}{2a_2}[1 - \sqrt{1 - 4(a_2/a_1)x}] = a_0 + \frac{2a_1x}{1 + \sqrt{1 - 4(a_2/a_1)x}}. \tag{20}$$

It is possible also to receive other variants of the function RG approximations, using other approaches for $f(x)$, $f^{'}(x)$, or $x(f)$ instead of the Taylor series . For example, one can get another RG approximation of the sought function using second convergent of the continued fraction corresponding to series (17):

$$f(x) \simeq f_{Xcf}^{(2)}(x) = a_0 + \frac{a_1x}{1 - (a_2/a_1)x} \tag{21}$$

It should be noted that this expression coincides with the approximate one obtained from (20) as a result of the square root expansion. It is interesting to note also the coincidence of this expression with the self-similar approximation [20].

Let us represent the estimations of the RG approximations accuracy of some functions for $x = 1$. We use the notation $\Delta_a = [f_a(1) - f(1)]/f(1) \times 100\%$, where $f_a(x)$ is any of the RG approximations of a function $f(x)$, as a measure of accuracy. For the function $f(x) = \ln(1 + x)$ we have $\Delta_\beta^{(2)} \approx -8.8$, $\Delta_\beta^{(3)} \approx 1.6$, $\Delta_X^{(2)} \approx 5.3$, $\Delta_{Xcf}^{(2)} \approx -3.8$, $\Delta_T^{(2)} \approx -27.8$, $\Delta_T^{(3)} \approx 20.2$, where the last two quantities refer to the finite Taylor series approximations of the square-law type and cubic one respectively. The accuracy of the different RG approximations for other function $f(x) = x/\sqrt{1+x}$ is represented as follows: $\Delta_\beta^{(2)} \approx -10.3$, $\Delta_\beta^{(3)} \approx 3.4$, $\Delta_X^{(2)} \approx 3.5$, $\Delta_{Xcf}^{(2)} \approx -5.4$, $\Delta_T^{(2)} \approx -28.9$, $\Delta_T^{(3)} \approx 24.0$.

It should be noted that the function RG approximations for $a_1 = 0$ can be obtained in the same way. In this case $x(f)$ is represented as a power series on $\phi_1 = \sqrt{(f - a_0)/a_2}$.

It is important to note, that the expansion in powers of x of both sought function $f(x)$ and its RG-approximation $f_a^{(n)}(x)$ coincide up to the order n inclusive. Therefore a successive RG-approximations $f_a^{(n)}(x)$ of the sought function $f(x)$ converge to the latter one not worse of the Taylor series while value n sequentially increases. RG-approximations of one-to-one function represent in general its better approximations in comparison with its Taylor series expansion because RG-approximations inherits its one-to-one nature.

IV. PARTITION FUNCTIONS OF STATISTICAL MECHANICS

Let us illustrate accuracy of our method on the two important physical examples. Both of them satisfy the following conditions: (a) the standard perturbation theory is not applicable at all leading to divergent series; (b) some analytical expressions are known; (c) analytical or numerical calculations are available giving the possibility to compare them with obtained results; and (d) there are other approximations that may also be compared with the method considered.

First example is an integral having the mathematical structure typical for the partition functions for statistical problems with effective potential of the form $V(\phi) = \phi^2 + g\phi^4$, that is, for the problems with the so-called ϕ^4 interaction. Consider the integral

$$I(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\phi^2 - g\phi^4)\, d\phi \tag{22}$$

with $g \in (0, \infty)$. The generating functional of the ϕ^4 quantum field theory has also the structure of (22).

Integral (22) admits an exact quadrature expressed via the modified Bessel function of the second kind:

$$I(g) = \frac{\exp(1/8g)}{\sqrt{4\pi g}} K_{1/4}(\frac{1}{8g}). \tag{23}$$

It is easy to find the expansion of (22) in powers of g:

$$I(g) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{\pi} n!} \Gamma\left(2n + \frac{1}{2}\right) g^n. \tag{24}$$

The strong coupling limit $g \gg 1$ of (22) can be obtain by using saddle-point method

$$I(g) \simeq \frac{1}{2\sqrt{\pi}} \left[\Gamma\left(\frac{1}{4}\right) g^{-1/4} - \Gamma\left(\frac{1}{4}\right) g^{-3/4}\right]; \qquad g \gg 1\,. \tag{25}$$

Substituting here the quantities $\Gamma(1/4) \approx 3.624\,409$, $\Gamma(3/4) \approx 1.225\,417$ for gamma-function $\Gamma(y)$, we get

$$I(g) \simeq 1.022\,735\, g^{-1/4} - 0.345\,384\, g^{-3/4}; \qquad g \gg 1\,. \tag{26}$$

As can see, the nth term in the expression (24) diverges at any non-zero g, as $n \to \infty$, since

$$\frac{\Gamma(2n + 1/2)}{n!} \simeq \left(\frac{4n}{e}\right)^n \to \infty\,, \qquad n \to \infty\,. \tag{27}$$

So, expansion in powers of g has no sense at $g \sim 1$. Retaining three terms of series (24), we get the approximation of (22)

$$I(g) \simeq 1 - \frac{3}{4}g + \frac{105}{32}g^2\,; \qquad g \ll 1\,. \tag{28}$$

The accuracy $\Delta_T^{(2)}$ of the approximation (28) is about 360 % at $g = 1$. On the other hand, we have $\Delta_\beta^{(2)} \approx -18\%$, $\Delta_X^{(2)} \approx 7\%$, and $\Delta_{Xcf}^{(2)} \approx 9\%$ for RG approximations (18), (20), and (21) respectively that are based on approximation (28). It is seems satisfactory accuracy for approximations based on a few terms of divergent series.

A. Modified RG approximation.

One can get more precise approximation by our method combined with the principle of minimal sensitivity (PMS) [21] even in the whole range $g \in (0, \infty)$.

Let us rewrite partition integral (22) in the equivalent form

$$I(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-z^2\phi^2 + \triangle V)\, d\phi, \tag{29}$$

where

$$\triangle V \equiv (z^2 - 1)\phi^2 - g\phi^4, \tag{30}$$

and z is trial parameter. Expanding the integrand of (29) in powers of (30), we get

$$I_1(g,z) \;=\; \frac{1}{z}\left[1 + \frac{1}{2}\left(1 - \frac{1}{z^2}\right) - \frac{3g}{4z^4}\right], \tag{31}$$

$$I_2(g,z) \;=\; I_1(g,z) + \frac{1}{z}\left[\frac{3}{8}\left(1 - \frac{1}{z^2}\right)^2 - \frac{15g}{8z^4}\left(1 - \frac{1}{z^2}\right) + \frac{105g^2}{32z^8}\right]. \tag{32}$$

in the first and in the second order respectively.

Let us make the substitution $x = 1 - 1/z^2$. Then expressions (31) and (32) transform into

$$I_1(g,x) \;=\; \sqrt{1-x}\left[1 + \frac{1}{2}x - \frac{3}{4}g(1-x)^2\right], \tag{33}$$

$$I_2(g,x) \;=\; I_1(g,x) + \sqrt{1-x}\left[\frac{3}{8}x^2 - \frac{15g}{8}x(1-x)^2 + \frac{105}{32}g^2(1-x)^4\right]. \tag{34}$$

Exact function $I(g)$ does not depend on z (or x) accordingly to (22) or (29), therefore $\frac{\partial I(g)}{\partial x} \equiv 0$. However, any finite series approximation $I_n(g,x)$ actually explicitly depends on x. According to the PMS [21], let us require minimal sensitivity to x as early as in the first order:

$$\frac{\partial I_1(g,x)}{\partial x} = 0\,, \tag{35}$$

that gives the relation

$$g = \frac{2}{5}\,\frac{x}{(1-x)^2}\,. \tag{36}$$

We see from here that $0 \le x \le 1$ owing to $0 \le g \le \infty$. Inverting this equation results in

$$x(g) = \frac{1 + 5g - \sqrt{1 + 10g}}{5g}\,. \tag{37}$$

Substituting (36) into (34), we obtain

$$I_2(x) = \sqrt{1-x}\left[1 + \frac{1}{5}x + \frac{3}{20}x^2\right]. \tag{38}$$

Finally, after substituting RG approximation (18) instead of the square brackets in (38), we get

$$I_\beta^{(2)}(g) = \sqrt{1-x(g)}\left\{1+\frac{2}{15}\left[\exp\left(\frac{3}{2}x\,(g)\right)-1\right]\right\}, \tag{39}$$

where $x(g)$ is given by formula (37). In accordance with the general property of the RG approximation method, the weak coupling limit ($g \ll 1$) of (39) up to the second order coincides with expansion (28). The strong coupling limit of (39), when $g \gg 1$, is

$$I_\beta^{(2)}(g) \simeq 1.164\,456\,g^{-1/4} - .634\,951\,g^{-3/4}; \qquad g \gg 1\,. \tag{40}$$

The accuracy of the approximation (39) is the monotone increasing function of g that tends to limit $\approx 13.8\%$ when $g \to \infty$.

B. Improved modified RG approximation.

It is important to note, that relation (36) was obtained by applying the PMS to the first order approximation (33) but not to the RG approximation. Therefore the relation (36) is not optimum for the RG approximation to achieve the best approximation of the exact function $I(g)$. Let us use more general relation instead of (36)

$$g = \frac{2}{5p}\frac{x}{(1-x)^2}\,. \tag{41}$$

with trial parameter p. Inverting this equation gives

$$x(g) = \frac{1+5pg-\sqrt{1+10pg}}{5pg}\,. \tag{42}$$

Using the relation (41) results in the following RG approximation

$$I_\beta^{(2)}(g) = \sqrt{1-x(g)}\left\{1+\frac{2}{15}\frac{\left[1+\frac{5}{2}(p-1)\right]^2}{1+\frac{5}{2}\,(p-1)^2}\left[\exp\left(\frac{3}{2}\frac{1+\frac{5}{2}\,(p-1)^2}{p\left[1+\frac{5}{2}\,(p-1)\right]}x\,(g)\right)-1\right]\right\}, \tag{43}$$

where $x(g)$ is given by formula (42).

The weak coupling limit of (43) again coincides with the expansion (28) up to the second order. As can be shown, this remarkable property is the consequence the fact that the modified perturbation potential $\triangle V$ (30) approaches the initial perturbation potential $g\phi^4$ when $x \to 0$.

On the other hand, the strong coupling limit of (43) depends on p. The optimal value of the parameter p can be found now from the coincidence of the strong coupling limit of $I_\beta^{(2)}(g)$ with this limit of the exact integral $I(g)$ that is given by (26), i.e.

$$\lim_{g\to\infty}\frac{I_\beta^{(2)}(g)}{I(g)} = 1 = \left(\frac{2}{5p}\right)^{1/4}\left\{1+\frac{2}{15}\frac{\left[1+\frac{5}{2}(p-1)\right]^2}{1+\frac{5}{2}\,(p-1)^2}\left[\exp\left(\frac{3}{2}\frac{1+\frac{5}{2}\,(p-1)^2}{p\left[1+\frac{5}{2}\,(p-1)\right]}\right)-1\right]\right\}\Big/1.022\,765\,. \tag{44}$$

Solving the last equality of (44) with respect to p we get

$$p \approx 1.779\,643. \tag{45}$$

Using this value, we get the following strong coupling limit of the RG approximation (43)

$$I_\beta^{(2)}(g) \simeq 1.022\,765\,g^{-1/4} - .343\,514\,g^{-3/4}, \qquad g \gg 1. \tag{46}$$

By numerical calculations, the maximal error for the RG approximation (43) with parameter p given by (45) is found to be $\approx 0.06\%$ at $g = 1.9$. Remind that when we did not optimized p ($p = 1$), the accuracy of the method was 13.8%. So, by using asymptotic constraint condition (44) we improved the accuracy more than in two hundred times.

V. QUARTIC ANHARMONIC OSCILLATOR

Now let us check the ability of our method to approximate the ground state energy E_0 of the one-dimensional anharmonic oscillator with the Hamiltonian ($\hbar = 1$)

$$H = -\frac{1}{2m}\frac{d^2}{dy^2} + \frac{m\omega^2}{2}y^2 + \lambda m^2 y^4\,, \tag{47}$$

in which m, ω and λ are positive constants, $y \in (-\infty\,,\ \infty)$.

Actually, a lot of various physical models can be reduced to the Hamiltonian (47). The close connection of the quartic-oscillator model with the so-called ϕ^4 model in the quantum field theory is also well acknowledged.

The Hamiltonian (47) , making the change $y \to \sqrt{m}y$, can be transformed to the form

$$H = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{\omega^2}{2}y^2 + \lambda y^4\,, \tag{48}$$

which is more convenient for calculations. The perturbation series with respect to the coupling constant λ diverges for any finite value of this constant [22, 23]. The divergence of the series for the dimensionless ground state energy

$$e(g) \equiv E_0/\omega \qquad (g \equiv \lambda/\omega^3) \tag{49}$$

is so strong that the expansion in powers of g has no sense at $g \sim 1$. This is due to the increase of coefficients near g^n according to $3^n n!$ as $n \to \infty$ [23]. The expansion of (49) in the weak-coupling limit [22, 23] is

$$e(g) \simeq \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \frac{333}{16}g^3 - \frac{30885}{128}g^4\,; \qquad (g \ll 1)\,. \tag{50}$$

The asymptotic behavior of (49) in the strong-coupling limit is given by [24, 25, 26]

$$e(g) \simeq g^{1/3}\left(0.667\,986 + 0.143\,67 g^{-2/3} - 0.008\,8 g^{-4/6}\right); \qquad (g \gg 1)\,. \tag{51}$$

To estimate our RG approximation, we take an advantage of exact numerical calculations [25, 26, 27] for the ground state energy (49) for a wide range of coupling constants. Retaining three terms of series (50), we get the approximation of (49), the relative accuracy $\Delta_T^{(2)}$ of which is about 270% at $g = 1$. In turn, RG approximations (18), (20), and (21) that based on this approximation give us $\Delta_\beta^{(2)} \approx 25\%$, $\Delta_X^{(2)} \approx -0.5\%$ and $\Delta_{Xcf}^{(2)} \approx 17\%$ respectively. This is rather reasonable accuracy for approximations based only on two first terms of divergent series.

A. Modified RG approximation.

Now we again improve our method to be applicable to arbitrary strong coupling. For doing this we take as a trial approximation the Hamiltonian

$$H_0 = -\frac{1}{2}\frac{d^2}{dy^2} + \frac{{\omega_0}^2}{2}y^2\,, \tag{52}$$

and calculate the ground state energy for (49) using the perturbation theory over $H - H_0 = (\omega^2 - {\omega_0}^2)y^2/2 + \lambda y^4$. In the zeroth order approximation we have $e_0(g, \omega_0) = \omega_0/2\omega$. The first approximation for (49) is

$$e_1(g, \omega_0) = \frac{\omega_0}{4\omega}\left[1 + \left(\frac{\omega}{\omega_0}\right)^2 + 3g\left(\frac{\omega}{\omega_0}\right)^3\right], \tag{53}$$

and the second one is

$$e_2(g, \omega_0) = e_1(g) - \frac{\omega_0}{16\omega}\left\{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2 - 6g\left(\frac{\omega}{\omega_0}\right)^3\right]^2 + 6g^2\left(\frac{\omega}{\omega_0}\right)^6\right\}. \tag{54}$$

Let us make the substitution $x = 1 - (\omega/\omega_0)^2$. Then expressions (53) and (54) become

$$e_1(g,x) = \frac{1}{2}\left[1 - \frac{1}{2}x + \frac{3}{2}(1-x)^{3/2}g\right](1-x)^{-1/2}\,, \tag{55}$$

$$e_2(g,x) = e_1(g,x) - \frac{1}{16}\left\{\left[x - 6\,(1-x)^{3/2}\,g\right]^2 + 6(1-x)^3 g^2\right\}(1-x)^{-1/2}\,. \tag{56}$$

Applying again the PMS [21] to the first order approximation (55)

$$\frac{\partial e_1(g,x)}{\partial x} = 0\,, \tag{57}$$

we get the following relation

$$g = \frac{1}{6}\,\frac{x}{(1-x)^{3/2}}\,. \tag{58}$$

We see from here that $0 \le x \le 1$ because of $0 \le g \le \infty$. Substituting (58) into (56) we get

$$e_2(x) = \frac{1}{2}\left[1 - \frac{1}{4}x - \frac{1}{48}x^2\right](1-x)^{-1/2}\,. \tag{59}$$

At last, replacing the square brackets in (59) by the RG approximation (18), we get

$$e_\beta^{(2)}(g) = \frac{1}{2}\left\{1 - \frac{3}{2}\left[\exp\left(\frac{1}{6}x\,(g)\right) - 1\right]\right\}[1 - x\,(g)]^{-1/2}\,. \tag{60}$$

Here $x(g)$ is positive and continuous solution of the equation (58) with respect to x:

$$x(g) = \begin{cases} 1 - \frac{3}{4}\,(g_c/g)^2\,[2\cos(\alpha/3) - 1]^2\,, & g \le g_c \\ 1 - \frac{3}{4}\,(g_c/g)^2\,[A_+ + A_- - 1]^2\,, & g \ge g_c \end{cases}\,, \tag{61}$$

where

$$\begin{aligned} g_c &= \frac{1}{9\sqrt{3}}\,, \\ \alpha &= \arccos\left(2\,[g/g_c]^2 - 1\right), \\ A_\pm &= \left\{-1 + 2\,(g/g_c)^2 \pm 2\,(g/g_c)\left[(g/g_c)^2 - 1\right]^{1/2}\right\}^{1/3}. \end{aligned} \tag{62}$$

In accordance with the method of the RG approximation, the week coupling limit of (60) coincides with the expansion (50) up to the second order . The strong coupling limit of (60), when $g \gg 1$, is

$$e_\beta^{(2)}(g) \simeq g^{1/3}\left(.661\,395 + .148\,035 g^{-2/3} - .010\,255 g^{-4/3}\right); \qquad (g \gg 1). \tag{63}$$

The accuracy of the approximation (60) monotone decreases with g and approaches a limit $\approx 0.99\%$ when $g \to \infty$.

B. Improved modified RG approximation.

To improve the accuracy we use again more general relation than (58)

$$g = \frac{1}{6p}\,\frac{x}{(1-x)^{3/2}}\,. \tag{64}$$

Using this relation results in the following RG approximation

$$e_\beta^{(2)}(g) = \frac{1}{2}\left\{1 - \frac{3}{2}\frac{(2p-1)^2}{1+6(p-1)^2}\left[\exp\left(\frac{1}{6}\frac{1+6(p-1)^2}{p\,(2p-1)}x\,(g)\right) - 1\right]\right\}[1 - x\,(g)]^{-1/2}\,, \tag{65}$$

where $x(g)$ is given by the same formula (61) as before but now $g_c = 1/(9\sqrt{3}p)$ instead of (62). The weak coupling limit of (65) coincide again with the expansion (50) up to the second order as in the previous section IV and on the same reasons. On the other hand, the strong coupling limit of (65) depends on p. The optimal value of the parameter p can be found now from the coincidence of the strong coupling limit of $e^{(2)}_\beta(g)$ with known exact limit of $e(g)$ which is given by (51):

$$\lim_{g\to\infty} \frac{e^{(2)}_\beta(g)}{e(g)} = 1 = \left(\frac{3p}{4}\right)^{1/3} \left\{1 - \frac{3}{2}\frac{(2p-1)^2}{1+6(p-1)^2}\left[\exp\left(\frac{1}{6}\frac{1+6(p-1)^2}{p\,(2p-1)}\right) - 1\right]\right\} \Big/ 0.667\,986\,. \tag{66}$$

Solving the last equality of (66) with respect to p we get

$$p \approx 1.472\,032. \tag{67}$$

With this value we get the following strong coupling limit of the RG approximation (65)

$$e(g) \simeq g^{1/3}\left(0.667\,986 + 0.143\,62 g^{-2/3} - 0.008\,6 g^{-4/6}\right)\,; \qquad (g \gg 1) \tag{68}$$

By numerical calculations, the maximal relative error for the RG approximation (65) with parameter p given by (67) is found to be $\Delta^{(2)}_\beta \approx 0.003\,6\%$ at $g = 0.7$. Remind that when we did not optimized p ($p = 1$), the accuracy of the method was 0.99%. Thus, we improved the accuracy in about three hundred times using asymptotic constraint condition (66).

VI. CONCLUSION

The method of RG approximation presented in this paper possesses the following peculiarities. It seems to be logical and simple. It needs minimal information, e. g., only two first terms of perturbation theory. Even in the simplest form the error of its approximation is about 10% at $g = 1$ even for a sought function that has divergent series in powers of g. The accuracy of the modified RG approximation that uses the PMS to make it applicable for arbitrary values of g is practically the same. The the accuracy of the approximation can be drastically improved in hundreds times while using the PMS if we take advantage of some fitting parameter to be defined by additional constraint condition, e.g., by making equal meaning of the sought function and its approximation at $g \to \infty$. So, the accuracy of the RG approximation of partition functions integral (22) is improved in more than two hundred times and is of 0.06%. The accuracy of the RG approximation of the ground state energy of the one-dimensional quartic anharmonic oscillator is also improved in the same way in almost three hundred times and is of $0.003\,6\%$.

The accuracy of our method can be compared with that of other analytical approaches that use an equivalent number of approximate terms, that is with the methods [20, 21, 28, 29, 30, 31]. The first two methods are called the modified perturbation theory, and the others are called the self-similar approximation. All of them also use either the PMS condition similar to (35), (57) or principle of minimal difference [20] to have an opportunity to approximate an interesting physical quantity in the whole range ($g \in 0, \infty$) of parameter g.

The accuracy of approximation of partition functions integral (22) by means of both the self-similar approximation [30] and the modified perturbation theory (38) is about 10%, practically the same as in our method. If in the latter one we use instead of (36) generalized relation (41) with the fitting parameter p being defined by asymptotic constraint condition similar to (44) then the accuracy of this approximation is improved up to 0.19%. As to the self-similar approximation, its accuracy is improved up to 1% by use of the "fixed-point distance" fitting parameter being also defined by asymptotic constraint condition. Remember that the method of improved modified RG approximation presented in this paper has the accuracy 0.06%.

In the case of calculation of the ground state energy of the one-dimensional anharmonic quartic oscillator the accuracy of the self-similar approximation is about 10% while using the principle of minimal difference [20] and is 0.3% if using the PMS [29, 31] (the accuracy cited in [29] corresponds to only strong-coupling limit). Different variants [21, 28, 32, 33, 34, 35] (see also formula (59)) of the second-order modified perturbation theory give the ground-state energy with the accuracy not higher than 1%. Notice that if we substitute in the second-order approximation (56)) instead of (58) the generalized relation (64) with the fitting parameter p being defined by asymptotic constraint condition similar to (66) then the accuracy of this approximation is improved up to 0.008%. Remember that the accuracy of our modified RG approximation (60) is about 1% whereas the accuracy of improved modified RG approximation (65) is $0.003\,6\%$.

Our approach allows a number of generalizations.

In general, the invariance relation (9) with respect to the RG transformation (12) is valid for any non-degenerate mapping of an associative set of elements x in other set. Indeed, relation (9) was obtained with the assumption that the variable value set x is a semigroup with respect to '+' operation. Evidently this set can be either continuous or discrete. In the latter case it is possible to represent the successive approximations of any function f as some discrete mapping. Let us use the number of the approach $(0, 1, 2, \ldots)$ as the argument of the discrete mapping and the successive approximation of the function $(f_0(x), f_1(x), f_2(x), \ldots)$ as the value of the mapping. In this case invariance relation (9) reads:

$$F(n + p; f_0(x)) = F(n; F(p; f_0(x))), \tag{69}$$

where n and p are non-negative integers. This relation (69) can be rewritten in the other form, if one use these discrete variables as the subscripts of the mapping F:

$$F_{n+p}(f_0(x)) = F_n(F_p(f_0(x))). \tag{70}$$

Such functional self-similar transformation was obtained [20] from other reasons and also was used for the constructing of the so called self-similar approximation.

Generally a mapping F connects elements x of one set M of the dimension m and elements f of another set N of the dimension n. In addition the RG invariance relation (9) is valid in the case of one-to-one correspondence between the image and preimage of the mapping. For the continuous sets and differentiable mappings this requirement is fulfilled, if the rank of the functional determinant $\partial(f_1, f_2, \ldots, f_n)/\partial(x_1, x_2, \ldots, x_m)$ is equal to the least of integers m and n (so called nondegenerate mapping). In the case of $m < n$ the nondegenerate mapping F defines some $m-$parametric nonintersecting subset in the set N (some $m-$dimensional nonintersecting hypersurface). The particular case $m = 1$ and $n = 2$ corresponds to the two-charge quantum-field model [3]. In this model the invariant charges obeys functional equations (3). In case of $m > n$ the mapping definition domain that is invariant with respect to RG transformation can be anyone $n-$dimensional nonintersecting hypersurface in M.

One should notice that the well known monotonous dependence of the invariant charge on the 4-momentum transfer squared in the gauge theories of the electro-weak and strong interactions that are renormalizable seems natural in the light stated above.

In this paper the RG approximation presented is based on the expansion of some function $f(x)$ in natural number powers of $(x - x_0)$ around some point x_0. However, our approach can be generalized to the case of both meromorphic functions that allow their expansion in positive and negative integer powers and arbitrary (nonmonotonic) functions. The latter generalization consists in representing of a real function as the real or imaginary part of some complex one-to-one function and RG approximating of the latter one.

These and others generalizations of our approach will be given in further publications.

The author would like to gratefully acknowledge the useful discussion with S.G. Rautian.

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